

Fast Parameter Estimation from the CMB Power Spectrum

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ABSTRACT

The statistical properties of a map of the primary fluctuations in the cosmic microwave background (CMB) may be specified to high accuracy by a few thousand power spectra measurements, provided the fluctuations are gaussian, yet the number of parameters relevant for the CMB is probably no more than about $10 - 20$. There is consequently a large degree of redundancy in the power spectrum data. In this paper, we show that the MOPED data compression technique can reduce the CMB power spectrum measurements to about 10-20 numbers (one for each parameter), from which the cosmological parameters can be estimated virtually as accurately as from the complete power spectrum. Combined with recent advances in the speed of generation of theoretical power spectra, this offers opportunities for very fast parameter estimation from real and simulated CMB skies. The evaluation of the likelihood itself, at Planck resolution, is speeded up by factors up to $\sim 10^8$, ensuring that this step will not be the dominant part of the data analysis pipeline.

Key words: cosmic background radiation - cosmology; theory - early Universe

1 INTRODUCTION

It has been recognised for roughly a decade that detailed study of the power spectrum of the fluctuations in the CMB could be used to obtain high precision values for several of the cosmological parameters, such as Ω_0 , H_0 and Ω_Λ (Bond & Efstathiou 1987, Kamionkowski, Spergel & Sugiyama 1994, Jungman et al. 1996). The physics of the CMB is much more straightforward than the complicated processes which affect the large-scale structure of the Universe, making it a much more promising laboratory for accurate parameter estimation. The main complications are the presence of foreground sources at microwave frequencies and proper accounting of instrumental noise effects, but recent balloon experiments, Boomerang (de Barnardis P. et al. 2000), MAXIMA (Hanany et al. 2000) and DASI (Pryke et al. 2001) have demonstrated that the main scientific goal is achievable with current technology. As experiments become more ambitious, the data processing requirements become more demanding, and the current datasets have sufficiently many pixels ($\sim 10^4 - 10^5$) that the data processing is already quite challenging. Even the first measurement of the CMB fluctuations, produced by the Cosmic Background Explorer (COBE) satellite (Smoot et al. 1992) produced a dataset with enough pixels (~ 4000) for data compression techniques to be valuable (Gorski 1994; Gorski K. et al. 1994; Bond 1995; Bunn & Sugiyama 1995). For the satellite experiments MAP (the Microwave Anisotropy Probe) and Planck (the Planck Surveyor Satellite), data compression will be vital. Each will provide very large datasets, with close to all-sky coverage with a resolution of up to 5 arcminutes, and $\sim 10^6 - 10^7$ pixels. The standard radical compression method is to reduce the map to a set of power spectrum estimates (see e.g. Bond, Jaffe & Knox 1998). In principle

this compression can be lossless, if the map is a gaussian random field (as closely predicted by inflation: see e.g. Gangui et al. 1994; Verde et al. 2000; Wang & Kamionkowski 2000), as all the statistical properties of the map are calculable from the power spectrum. The power spectrum data, typically a few thousand numbers for a high-resolution experiment, can then be used to estimate cosmological parameters to an accuracy of a few percent. The steps in the distillation of the raw data to the cosmological parameters are, however, not necessarily straightforward computationally (see e.g. Wright 1996; Muciaccia, Natoli & Vittorio 1997; Tegmark 1997a; Tegmark 1997b; Bond et al. 1999; Olive, Spergel & Hinshaw 1999; Borrill 1999; Wandelt, Hivon & Gorski 2000; Szapudi et al. 2001; Natoli et al. 2001; Hivon et al. 2001; Christensen et al. 2001). This paper addresses one aspect of this problem: parameter estimation from the power spectrum. MOPED^{*} is an eigenvector-based method for data compression and parameter estimation, originally developed for computing star-formation histories from galaxy spectra (see Heavens, Jimenez & Lahav 2000, hereafter HJL; Reichardt, Jimenez & Heavens 2001). It can also be employed for very accurate, and extremely fast, parameter estimation from the CMB. The speed-up over brute-force maximum likelihood method is dependent on the experiment: typical speed-up factors expected for MAP and Planck are between 10^7 and 10^9 . MOPED is much more powerful than necessary, in fact, as parameter estimation will be dominated by the time it takes to run predictions for cosmological models, or other steps in the analysis pipeline.

The method is based on a technique developed by HJL for

^{*} MOPED (Multiple Optimised Parameter Estimation and Data Compression) has patent protection

compressing and analysing galaxy spectra. In that paper, it was shown that datasets with certain noise properties offered possibilities for very radical linear compression of the data without any loss of information about the parameters which determine the data. The requirement is for a dataset whose mean depends on the parameters, but the covariance of the noise does not. In these circumstances, it is possible to find a set of linear combinations of the data which are *locally sufficient statistics* for the parameters - i.e. the compressed data contain as much information about the parameters as the full dataset, and in this sense the compression is lossless (strictly, the Fisher matrix is unchanged, so the likelihood surface is known to be unchanged only locally near the peak). The compressed dataset can be extremely small - it consists of a single number for each parameter. Thus for highly redundant datasets, the degree of compression can be very large.

It is important to recognise that the data compression can still be done even if the assumptions for lossless compression do not apply. The main assumptions are that the information is contained in the mean of the data, not in their variance, and that the fiducial model is correct. Violation of neither of these is serious for CMB power spectrum analysis. In HJL, for example, the data compression algorithm was applied to the case of galaxy spectra, where the noise includes a photon counting noise term which is dependent on the mean number of photons in the spectral channel, and hence does depend on the parameters of the galaxy. The compressed data can still be used for parameter estimation, but the error bars on the derived parameters are fractionally larger than by using the full spectrum. The same situation arises in the CMB: under general assumptions, the cosmic variance on a power measurement is proportional to the square of the power itself, and therefore is dependent on the underlying parameters. The data compression, although not lossless, is still highly efficient: conditional errors should increase by a factor $\sim 1/\ell_{\max}$ for an experiment measuring multipoles up to ℓ_{\max} . The time required for a brute-force likelihood evaluation is broadly comparable to the time it takes to compute theoretically the power spectrum of a model, using CMBFAST (Seljak & Zaldarriaga 1996). Significantly, this part of the process has been accelerated recently by a factor of $\sim 10^3$ (Tegmark, Zaldarriaga & Hamilton 2001), making it much faster to compute the theoretical power spectra than computing a brute-force likelihood measurement. The relative timings for these two steps can determine the analysis strategy, since if the computation of the theoretical power spectrum is small in comparison with the likelihood evaluation, one can calculate the power spectrum ‘on the fly’ as one searches through parameter space. A useful goal is therefore to make the likelihood evaluation much quicker than computation of the theoretical power spectrum. One can already speed up this process by using variants of the Newton-Raphson method (see, e.g. Bond et al. 1999), and one can argue that the power of MOPED is not strictly necessary for this problem. However, it is possible that calculations of theoretical power spectra will be accelerated still further, but this paper shows that, with MOPED, the analysis need never be dominated by likelihood evaluations.

In this paper, we demonstrate that MOPED does successfully recover cosmological parameters from simulated datasets, but many orders of magnitude more quickly. We also show that the parameter errors are similar to the full maximum likelihood solution.

2 MASSIVE LOSSLESS DATA COMPRESSION

The method is detailed in HJL, so we only sketch details here. We define the data vector \mathbf{x} as the estimates of the power spectrum $\{\hat{C}_\ell\}$, where ℓ is the angular multipole, in terms of signal C_ℓ and noise n_ℓ :

$$\hat{C}_\ell = C_\ell(\theta_\alpha) + n_\ell \quad (1)$$

where θ_α are the set of cosmological parameters on which the CMB power spectrum depends. The noise is assumed to have zero mean, so

$$\langle \hat{C}_\ell \rangle = C_\ell(\theta_\alpha) \quad (2)$$

and the noise covariance matrix, including cosmic variance and instrument noise, is $\mathcal{N}_{\ell\ell'} = \langle n_\ell n_{\ell'} \rangle$. Angle brackets indicate ensemble averages; these are calculable analytically for some algorithms of power spectrum estimation (see e.g. Tegmark 1997b), but for others, e.g. based on correlation functions (Szapudi et al. 2001), a Monte Carlo approach is required. In practice this should be the covariance of the *estimates* of the power spectrum. Since this is dependent on the algorithm used to estimate the power spectrum, we assume for illustration only cosmic variance, modelled as gaussians with variance $\frac{2C_\ell^2}{(2\ell+1)}$, but in addition we do correlate the power spectrum estimates to mimic partial sky coverage. This approximation may not be good, especially for low multipoles. Bond, Jaffe & Knox (2000) have argued that the distribution may be closer to an offset lognormal, in which case one can transform the power spectrum estimates to quantities which have nearly gaussian marginal distributions. The calculation we show is illustrative, but Planck will be cosmic variance limited up to high ℓ .

The brute force maximum likelihood method, which uses all the power spectrum data points, is the method of estimation which for a large dataset will provide the smallest errors, assuming uniform priors. The likelihood for the N parameters is

$$\mathcal{L}(\theta_\alpha) = \frac{1}{(2\pi)^{\frac{N}{2}} \sqrt{|\mathcal{N}|}} \exp \left\{ -\frac{1}{2} \sum_{\ell\ell'} [\hat{C}_\ell - C_\ell(\theta_\alpha)] \mathcal{N}_{\ell\ell'}^{-1}(\theta_\alpha) [\hat{C}_{\ell'} - C_{\ell'}(\theta_\alpha)] \right\} \quad (3)$$

The difficulty is that at each point in parameter space one generally computes the determinant of, and inverts, an $N \times N$ matrix. Since this scales as N^3 , it becomes a significant computational expense, even with $N \simeq 2000$. In this context, significant means that it exceeds significantly the time taken currently to generate the theoretical power spectrum.

We can speed up the likelihood evaluation by using MOPED to compress the N data in the measured \hat{C}_ℓ to one datum for each of M unknown parameters. The algorithm is detailed in HJL; it produces a set of weighting vectors \mathbf{b}_α ($\alpha = 1 \dots M$), from which a set of MOPED components

$$y_\alpha \equiv \mathbf{b}_\alpha \cdot \mathbf{x} \quad (4)$$

is constructed. The MOPED vectors are designed to make the Fisher information matrix

$$F_{\alpha\beta} \equiv - \left\langle \frac{\partial^2 \ln \mathcal{L}}{\partial \theta_\alpha \partial \theta_\beta} \right\rangle \quad (5)$$

the same whether we use the compressed data y_α or the full set of power spectrum estimates. In fact this is only possible if we ignore the dependence of cosmic variance on the parameters, but this

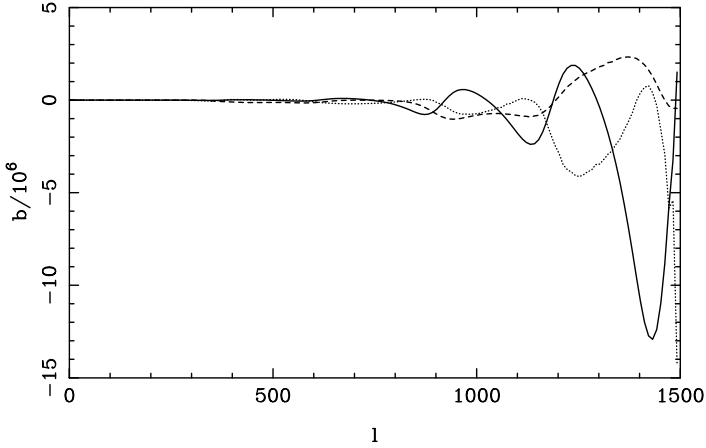


Figure 1. Optimised MOPED weighting vectors for a fiducial model with $H_0 = 65 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_{CDM} = 0.254$ and $\Omega_\Lambda = 0.7$. The parameter ordering (see text) is Ω_Λ , H_0 and Ω_{CDM} , and the MOPED vectors \mathbf{b}_1 , \mathbf{b}_2 , \mathbf{b}_3 , are shown by the solid, dashed and dotted lines respectively. Derivatives of the power spectrum have been calculated using finite-differences, which can cause the small glitches seen in this figure.

restriction makes virtually no difference for a CMB dataset. The MOPED vectors satisfy the following (HJL equation 14):

$$\mathbf{b}_{\ell 1} = \frac{\mathcal{N}_{\ell\ell'}^{-1} \frac{\partial C_{\ell'}}{\partial \theta_1}}{\frac{\partial C_{\ell\ell'}}{\partial \theta_1} \mathcal{N}_{\ell'\ell''}^{-1} \frac{\partial C_{\ell''}}{\partial \theta_1}} \quad (6)$$

$$\mathbf{b}_{\ell\alpha} = \frac{\mathcal{N}_{\ell\ell'}^{-1} \frac{\partial C_{\ell'}}{\partial \theta_\alpha} - \sum_{\beta=1}^{\alpha-1} \left(\frac{\partial C_{\ell\ell'}}{\partial \theta_\alpha} \mathbf{b}_{\beta\ell'} \right) \mathbf{b}_{\beta\ell}}{\sqrt{\frac{\partial C_{\ell\ell'}}{\partial \theta_\alpha} \mathcal{N}_{\ell'\ell''}^{-1} \frac{\partial C_{\ell''}}{\partial \theta_\alpha} - \sum_{\beta=1}^{\alpha-1} \left(\frac{\partial C_{\ell\ell'}}{\partial \theta_\alpha} \mathbf{b}_{\beta\ell'} \right)^2}} \quad (7)$$

and the summation convention is assumed. $\mathbf{b}_{\ell\alpha}$ refers to the ℓ component of the vector labelled by α . Obvious modifications are made if the data does not include all ℓ values - the vector components refer to the list of modes considered. Note that the MOPED vectors depend on the order in which the parameters are listed: y_1 contains as much information about parameter 1 as possible. This vector also constrains parameter 2 to some extent; y_2 adds as much additional information as possible about parameter 2, etc. A set of 3 MOPED vectors is illustrated in Fig. 1, corresponding to vacuum energy density, Hubble constant and cold dark matter (CDM) density. These vectors would ensure, under certain assumptions, that the MOPED components y_α are uncorrelated, and of unit variance; if this is the case, the likelihood with these as the data is simply

$$\mathcal{L}(\theta_\alpha) = \frac{1}{(2\pi)^{3/2}} \exp \left[-\frac{1}{2} \sum_{i=1}^3 (y_i - \langle y_i \rangle)^2 \right] \quad (8)$$

where the $\langle y_i \rangle$ are computed from the noise-free (but smoothed) theoretical power spectra. Importantly, they ensure that the Fisher matrix for the compressed dataset $\{y_\alpha\}$ is the same as for the entire set of power spectrum estimates. The marginal error on a single parameter is $[(F^{-1})_{\alpha\alpha}]^{\frac{1}{2}}$ and the error on the parameter estimated using any method cannot be smaller than this (see e.g. Kendall & Stuart 1969; Tegmark, Taylor & Heavens 1997). Thus, by ensuring that the Fisher matrices coincide, the compression method can be described as locally lossless - the parameter errors, as estimated

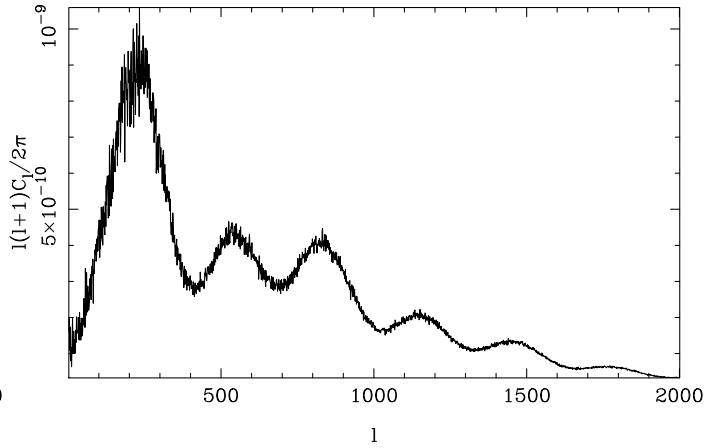


Figure 2. Simulated realisation of the CMB power used in the analysis.

from the local curvature of the likelihood surface at the peak, are on average no larger for the compressed data than for the full set of power spectrum estimates.

In detail, the assumptions required for locally lossless compression do not hold for this analysis. In order to calculate the MOPED vectors, the data covariance matrix, and the derivatives of the power spectrum with respect to the parameters, need to be known. These are fixed by assuming a fiducial set of parameters. We show below that this fiducial set is not important, but one can iterate the process if desired, at minimal extra computational expense. Our results show that iteration is actually unnecessary. The second assumption is that the covariance matrix of the data is not dependent on the model parameters. This is not strictly true for the CMB power spectrum, as the noise includes a cosmic variance term which is dependent on the cosmology. However, this does not prevent us compressing the data, and, in fact the Fisher matrix is dominated by the sensitivity of the power spectrum itself to the parameters, rather than the sensitivity of the noise.

A few remarks on speed are in order. With N power spectrum estimates, brute-force likelihood calculations require $O(N^3)$ calculations. With M parameters, MOPED requires $O(M)$ operations per likelihood evaluation. In addition, there are $O(MN)$ operations to compute the $\langle y_i \rangle$ quantities, but these can be done in advance if a library of theoretical power spectra is built up prior to analysis of the data. This point is potentially important for Planck; libraries of theoretical power spectra (and $\langle y_i \rangle$) can be constructed in the years before launch; if so, the parameter estimation step can be a very fast process, utilising interpolation of the $\langle y_i \rangle$ if desired.

In addition to this, there is a one-off $O(MN^3)$ operation to compute the MOPED vectors. The number of likelihood evaluations required to find the maximum is not easy to compute a priori, but is likely to depend exponentially on M , so for a large-dimensional parameter space, the overhead in computing the vectors is negligible in comparison with time spent in searching the space.

3 RESULTS

We simulate a CMB dataset by adding gaussian noise, at the level of cosmic variance, to theoretical power spectra produced by CMB-

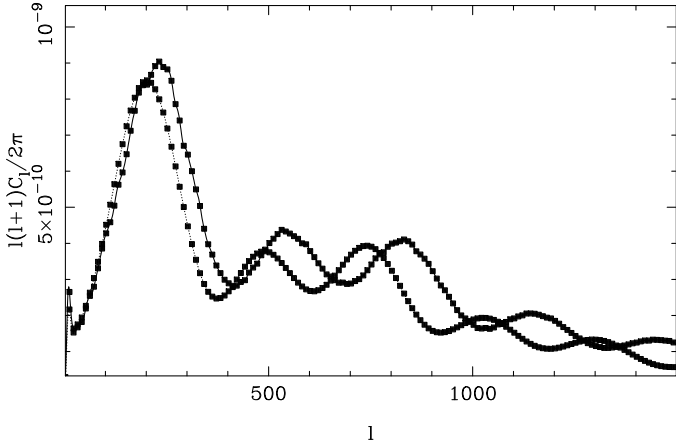


Figure 3. The true model spectrum (solid), with $H_0 = 65 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_\Lambda = 0.7$ and $\Omega_{\text{CDM}} = 0.254$, with gaussian noise and smoothed in ℓ with a gaussian of width $\Delta\ell = 5$. Also shown (dotted) is the fiducial model used in the data compression for fig.5: $H_0 = 69.8 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_\Lambda = 0.758$ and $\Omega_{\text{CDM}} = 0.254$, both smoothed with a gaussian of width $\Delta\ell = 5$. The boxes show the data points used for the likelihood calculations.

FAST. The power spectrum is convolved with a gaussian of chosen width, to mimic approximately the correlations in power spectrum estimates introduced by partial sky coverage. The dataset consisted of the power spectrum sampled in even steps in ℓ . The model chosen has $H_0 = 65 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_\Lambda = 0.7$ and $\Omega_{\text{CDM}} = 0.254$. The unconvolved power spectrum is shown in fig. 2, and the convolved spectrum in fig. 3.

We calculate the full (equation 3) and compressed (equation 8) likelihoods, varying the calculation in the following ways:

- We mimic the effects of partial sky coverage by convolving the power spectrum with a gaussian window function of various widths; we present results for a width of $\Delta\ell = 5$.
- The size of the dataset N is varied by changing the upper multipole limit of the available data, or by missing out some C_ℓ values.
- We explore different fiducial models, to see if the method is sensitive to an accurate initial guess of the parameters.

We fix most of the cosmological parameters. The values are not particularly important, but are listed here: $\Omega_B = 0.05$; scalar spectral index $n = 1$; no tensor modes; no massive neutrinos; 3 massless neutrinos. The parameters we allow to vary are the vacuum energy density parameter Ω_Λ , the CDM density parameter Ω_{CDM} and the Hubble constant H_0 , although we only display likelihood surfaces in the $\Omega_\Lambda - H_0$ plane, with Ω_{CDM} fixed.

Figure 4 shows the $H_0 - \Omega_\Lambda$ likelihood surface using the power spectrum of Figure 2 up to $\ell = 1500$ in steps of 10. The power estimates were smoothed with a gaussian of width 5. The calculation of this grid of likelihoods took 9463 seconds of CPU on an alpha workstation. Figure 5 shows the likelihood using 3 MOPED components as compressed data. An incorrect fiducial model ($H_0 = 69.8 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_\Lambda = 0.758$, $\Omega_{\text{CDM}} = 0.254$) was chosen, to illustrate that its choice is not important. The true solution is still recovered accurately, but much faster: 0.00098 seconds, or an improvement of order 10^7 .

In order to check that the compressed data recover the parameters as accurately as the full data, we degrade the experiment, trun-

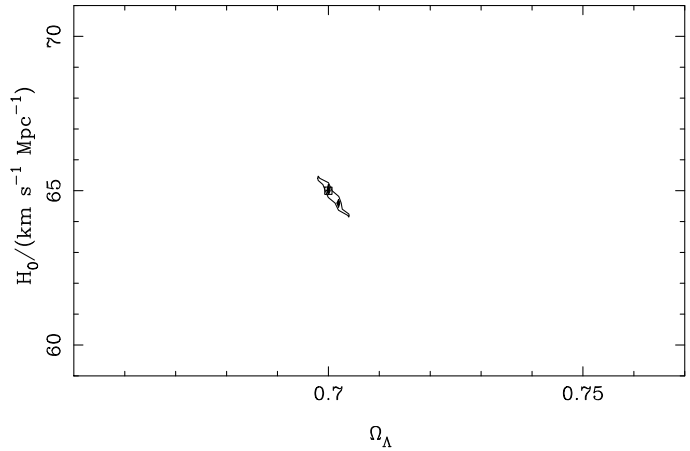


Figure 4. Likelihood surface for Ω_Λ and H_0 obtained from the the full dataset. This dataset consists of 150 power spectrum estimates from $\ell = 2, \dots, 1500$ in steps of 10, smoothed over a scale of $\Delta\ell = 5$. The true model is labelled with a square. The likelihood contours are too small to see individually for this experiment; the outer contour contains 99.99% of the probability, assuming uniform priors.

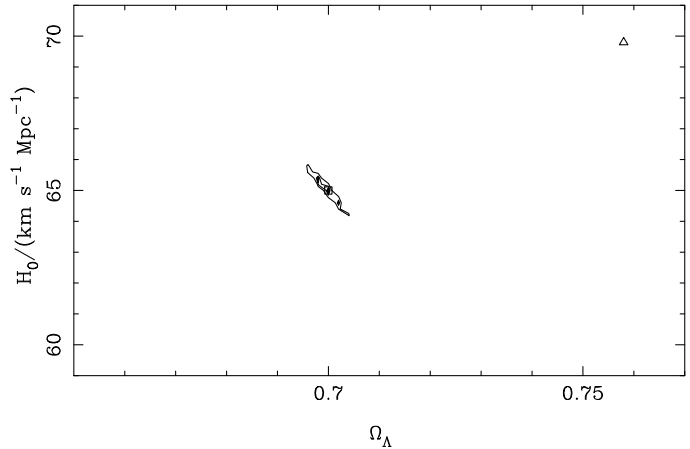


Figure 5. Likelihood surface for Ω_Λ and H_0 obtained from the the 3 MOPED components. The fiducial model used for the data compression no longer coincides with the true model, and is marked by a triangle. Note that the method still recovers the correct model (square).

ating the data to $\ell = 2, \dots, 300$, in steps of 10 (fig. 6 and 7). The method is designed to ensure that the error bars should be almost the same as the full likelihood on average, and we see that for this realisation the errors are indeed comparable. The full likelihood calculation takes 1406 seconds, while MOPED takes 0.00016 seconds. We see here that with a very poor fiducial model, MOPED still correctly finds the solution, within the errors, but there is a suggestion that the errors are only approximately correct. This can arise because the y_i are assumed to be uncorrelated, and this is only strictly true if the fiducial model is correct, and even then it is only the ensemble average errors which are unchanged. In practice, this is not a problem, as we have a much better idea now of the shape

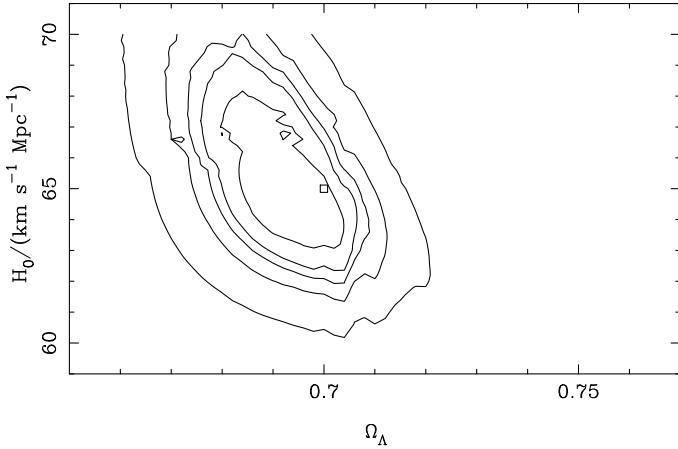


Figure 6. Likelihood from the full power spectrum, as in fig. 4, but restricted to $\ell \leq 300$ in steps of 10, to illustrate the size of the error bars. The contours represent confidence limits of 99.99%, 99%, 95.4%, 90%, and 68%. The true model is labelled with a square. The likelihood was calculated on a grid covering $60 \leq H_0 \leq 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ in steps of 0.2, and $0.66 \leq \Omega_\Lambda \leq 0.76$ in steps of 0.002.

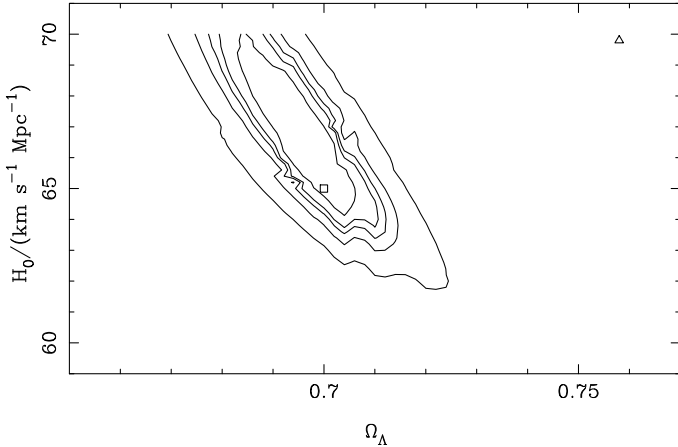


Figure 7. As fig. 6, but showing the likelihood from MOPED components. Note that the error bars are comparable.

of the power spectrum, so can choose a fiducial model which is far better than this one. Secondly, one can iterate, at very modest extra computational expense, computing new MOPED vectors from the best previous estimate.

4 CONCLUSIONS

The steps required to turn a set of power spectrum measurements C_ℓ into estimates of cosmological parameters consist of

- Computation of theoretical C_ℓ
- Calculation of likelihood of model parameters
- Maximisation of likelihood and marginalisation

Tegmark, Zaldarriaga & Hamilton (2001) have addressed the speed of the first step, accelerating CMBFAST (Seljak & Zaldarriaga 1996) by a factor $\sim 10^3$. This paper complements that analysis by speeding up the brute-force likelihood evaluation in the second step by even larger factors. For N correlated data points, a brute-force likelihood evaluation using all the data scales as N^3 . MOPED reduces this to M approximately uncorrelated, unit variance components, whose likelihood evaluation scales with the number of parameters M . For a Planck-size dataset with $N = 2000$ and $M \sim 12$ parameters, the speed-up factor should be around 500 million. In a sense MOPED is much more powerful than it needs to be, but this is hardly a criticism. With MOPED and the advances of Tegmark, Zaldarriaga & Hamilton (2001), parameter estimation is accelerated by a useful factor of $\sim 10^3$, and we can be fairly certain that the data processing element will be dominated by other steps in the analysis pipeline.

The speed of MOPED may influence the analysis strategy; if the likelihood evaluation is slow in comparison with theoretical power spectrum generation, then one can compute the power spectra ‘on-the-fly’ in a search for the maximum likelihood. Given that the position is now reversed, there is a case for creating grids of theoretical models in the years before launch of Planck. If storage space becomes a limiting issue, one can store the expected MOPED components for each model, rather than the full C_ℓ , with a compression factor > 100 . However, there may well still be a case for less rigid searches of parameter space, such as Markov Chain Monte Carlo methods (Christensen et al. 2001), since they can simultaneously estimate the shape of likelihood surface around the peak, as well as finding the peak itself. MOPED can be combined with such methods to advantage. Finally we note that, for current experiments, data compression is not necessary, as there are relatively few band-power estimates available.

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